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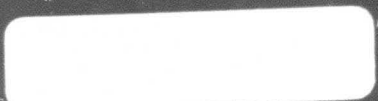
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AN EXACT LOWER CONFIDENCE BOUND FOR THE RELIABILITY OF A SERIES SYSTEM  
WHERE EACH COMPONENT HAS AN EXPONENTIAL TIME TO  
FAILURE DISTRIBUTION

by  
Tapas K. Sarkar

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AN EXACT LOWER CONFIDENCE BOUND FOR THE RELIABILITY OF A SERIES SYSTEM  
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1. Introduction and Summary

Let us consider a complex system consisting of  $k$  components. The components are connected in series so that the system fails if at least one component fails. Let  $T_i$  represent the time to failure of the  $i^{\text{th}}$  component. The reliability of the system at time  $t$  is given by

$$(1) \quad R(t) = P[T_1 > t, T_2 > t, \dots, T_k > t].$$

This implies that  $T$ , the time to failure of the system, is given by

$$(2) \quad T = \min_{i=1, \dots, k} [T_i] \text{ and } R(t) = P[T > t].$$

Let us assume that  $T_i$  has an exponential distribution with parameter  $\lambda_i$  so that

$$(3) \quad P(T_i > t) = e^{-\lambda_i t}, \quad i = 1, \dots, k.$$

Initially, let us assume that  $T_1, \dots, T_k$  are independently distributed. Then from (1) and (2),

$$(4) \quad R(t) = P[T > t] = e^{-(\lambda_1 + \dots + \lambda_k)t},$$

so that the time to failure of the system has an exponential distribution with parameter  $\lambda_1 + \lambda_2 + \dots + \lambda_k$ . In general,  $\lambda_1, \dots, \lambda_k$  will not be

known. We want to find a lower confidence bound for  $R(t)$ .

We assume that the system as a whole is either impossible or too expensive to test. Hence, no data is available for  $T$ . However, one can test each individual component and observations are available for  $T_i$  for each  $i = 1, \dots, k$ . We want our confidence bound to be based on these latter observations.

Kraemer [7] has given two methods for obtaining the lower confidence bound of the above reliability. However, her methods do not give an exact lower confidence bound. The method proposed by us will give an exact lower confidence bound. It will also be shown that our method works under certain circumstances even when  $T_1, \dots, T_k$  are not independently distributed.

We have given numerical examples to compare Kraemer's first method with ours. We have also compared our lower confidence bound with that given by maximum-likelihood estimates.

Kraemer's second method is based on a version of Tchebyshev's inequality. It appears to us that in most cases it is inferior to Kraemer's first method. Accordingly, we will not mention it any further.

Lentner and Buehler [8] have used the Lehmann-Scheffé theory to obtain a uniformly most powerful unbiased test for the hypothesis  $\lambda_1 + \lambda_2 = \phi$ . Inverting the test they obtained an exact confidence bound for the reliability of the system. The method has been extended to the case of  $k$  components by El Mawaziny [1]. Unfortunately, the methods are complicated and even in the case of  $k = 2$  and very small sample sizes one has to solve equations by trial and error to obtain the confidence bounds for  $\phi$ . For larger values of  $k$  and/or for moderate sample sizes elaborate computer programming is required. In fact, El Mawaziny and

Buehler [2] derived an approximate method when there are  $k(k \geq 2)$  components because of the difficulty in computation for their exact methods. It will be shown below that our method is very easy to compute and can be easily worked out with a desk calculator. The methods of [8] and [1] are applicable only when the components are independent. As will be shown, our method also works for dependent components.

El Mawaziny and Buehler have used a theorem of R. von Mises to obtain large sample approximations of the methods of [8] and [1]. Their method depends on asymptotic expansions of certain quantities. It turns out that if the first two terms of the expansions are used (and the higher order terms are ignored), the confidence bound of reliability is the same as that given by the maximum-likelihood method. The approximation can be improved by using higher order terms, but the degree of improvement is not known.

## 2. The Method

To illustrate the method, let us assume that  $k = 2$ .

Let  $T_1$  be independent exponentially distributed random variables with parameters  $\lambda_i$ ,  $i = 1, 2$ . Let  $T = \min(T_1, T_2)$ . Then

$$P(T > t) = P(T_1 > t, T_2 > t) = e^{-(\lambda_1 + \lambda_2)t}.$$

Hence,  $T$  is exponentially distributed with parameter  $(\lambda_1 + \lambda_2)$ .

Let us assume that  $n_i$  similar components of type  $i$  are on test simultaneously. The test is without replacement so that as one component fails it is removed from test and is not replaced. We terminate

the test as soon as  $r_i$  among  $n_i$  have failed. This method is known in the literature as type II censoring of  $r_i$  out of  $n_i$  without replacement.

Let  $X_{(1)} < X_{(2)} < \dots < X_{(r_1)}$  be the order statistics of the times to failure of the components of the first type, and  $Y_{(1)} < Y_{(2)} < \dots < Y_{(r_2)}$  be the same for the components of the second type. For our method we require that  $r_1 = r_2$ . Let  $r = \min(r_1, r_2)$ . Hence, we work with termination points  $r$  for each component. Kraemer's method or the methods of [1], [2], and [8], however, do not require this restriction. For simplicity, let us assume that  $n_1 = n_2 = n$ . As will be evident, there is no loss of generality in making this second assumption.

Let

$$\begin{aligned} D_1 &= X_{(1)} \\ (5) \quad D_j &= X_{(j)} - X_{(j-1)}, \quad j = 2, \dots, r. \\ \delta_j &= (n-j+1)D_j, \quad j = 1, \dots, r. \end{aligned}$$

Epstein and Sobel [3], and Epstein [4] have shown that  $\delta_1, \dots, \delta_r$  are independently, identically distributed. The distribution is exponential with parameter  $\lambda_1$ . Similarly, let

$$\begin{aligned} E_1 &= Y_{(1)} \\ (6) \quad E_j &= Y_{(j)} - Y_{(j-1)}, \quad j = 2, \dots, r. \\ \tau_j &= (n-j+1)E_j, \quad j = 1, \dots, r. \end{aligned}$$

Then  $\tau_1, \dots, \tau_r$  are independently exponentially distributed with parameter

$\lambda_2$ . Also,  $\tau_1, \dots, \tau_r$  are independent of  $\delta_1, \dots, \delta_r$ .

Let

$$(7) \quad z_j = \min(\delta_j, \tau_j).$$

Then, as we have already stated,  $z_1, \dots, z_r$  are independently exponentially distributed with parameter  $\lambda_1 + \lambda_2$ . Let

$$(8) \quad rV = \sum_{j=1}^r z_j.$$

It is well known that  $2rV(\lambda_1 + \lambda_2)$  has chi-square distribution with  $2r$  degrees of freedom. Let  $\chi^2$  denote a random variable distributed as chi-square with  $2r$  degrees of freedom. Let  $\chi_{2r}^2(\alpha)$  be such that

$$(9) \quad P[\chi^2 < \chi_{2r}^2(\alpha)] = 1 - \alpha.$$

Hence,

$$P[2rV(\lambda_1 + \lambda_2) < \chi_{2r}^2(\alpha)] = 1 - \alpha$$

or

$$P[\lambda_1 + \lambda_2 < \frac{\chi_{2r}^2(\alpha)}{2rV}] = 1 - \alpha$$

and, since  $R(t) = e^{-(\lambda_1 + \lambda_2)t}$  is a decreasing function of  $\lambda_1 + \lambda_2$  for fixed  $t$ , we get

$$(10) \quad P[e^{-(\lambda_1 + \lambda_2)t} > \exp\left\{-t \cdot \frac{\chi_{2r}^2(\alpha)}{2rV}\right\}] = 1 - \alpha.$$

This gives an exact  $100(1 - \alpha)$  per cent lower confidence bound on the reliability of the system at time  $t$  when the system consists of two components in series and the time to failure distribution of each component is exponential and the two exponentials are independent.

It is obvious that the above method does not require the assumption



that  $n_1 = n_2 = n$ . If  $n_1$  and  $n_2$  are unequal, then in (5) and (6) the definition of  $\delta_j$  and  $\tau_j$  will be changed as follows:

$$(11) \quad \left. \begin{aligned} \delta_j &= (n_1 - j + 1)D_j \\ \tau_j &= (n_2 - j + 1)E_j \end{aligned} \right\} j = 1, 2, \dots, r.$$

The rest of the theory goes through and (7), (8), (9), and (10) hold with the new definitions of  $\delta_j$ ,  $\tau_j$ , and  $z_j$ .

### 3. Extensions of the Method

#### A. When the number of components is larger than 2.

Let us consider a system consisting of  $k$  components in series. Let  $T_i$  be the time to failure of the  $i^{\text{th}}$  component. Assume that  $T_1, T_2, \dots, T_k$  are independent exponentially distributed random variables having parameters  $\lambda_1, \lambda_2, \dots, \lambda_k$  respectively.

Let  $T = \min(T_1, \dots, T_k)$ . Then

$$P(T > t) = P(T_1 > t, \dots, T_k > t).$$

Since  $T_1, T_2, \dots, T_k$  are independent,

$$P(T > t) = P(T_1 > t) \dots P(T_k > t) \\ = e^{-(\lambda_1 + \dots + \lambda_k)t}.$$

Hence,  $T$  is exponentially distributed with parameter  $\lambda_1 + \dots + \lambda_k \equiv \lambda$ , say.

Suppose we have  $n$  components of each type on test, and let us perform type II censoring of  $r$  out of  $n$  without replacement. Let

our data consist of  $X_{(1)}^{(i)} < X_{(2)}^{(i)} < \dots < X_{(r)}^{(i)}$ ,  $i = 1, \dots, k$ . As in section 2, for  $i = 1, 2, \dots, k$ , let

$$\begin{aligned} D_1^{(i)} &= X_1^{(i)} \\ (12) \quad D_j^{(i)} &= X_{(j)}^{(i)} - X_{(j-1)}^{(i)}, \quad j = 2, \dots, r, \\ \delta_j^{(i)} &= (n-j+1)D_j^{(i)}, \quad j = 1, \dots, r, \\ (13) \quad z_j &= \min_{i=1, \dots, k} \delta_j^{(i)}, \quad j = 1, \dots, r. \end{aligned}$$

$z_1, \dots, z_r$  are independently exponentially distributed with parameter  $\lambda$ .

Let

$$(14) \quad rV = \sum_{i=1}^r z_i.$$

Then  $2rV\lambda$  has chi-square distribution with  $2r$  degrees of freedom, and

$$(15) \quad P[e^{-\lambda t} > \exp \left\{ -t \cdot \frac{\chi_{2r}^2(\alpha)}{2rV} \right\}] = 1 - \alpha.$$

This gives us an exact  $100(1-\alpha)$  per cent lower confidence bound for the reliability of this system at time  $t$ .

If  $n_1, \dots, n_k$  are different, then the modification proposed in the last paragraph of section 2 applies.

**B. When  $k = 2$  and the distributions are not independent.**

Marshall and Olkin [9] and Harris [6] have considered the following model of a bivariate exponential distribution.

Let us consider 3 independent series of events happening in time. Let us consider the series  $i$ ,  $i = 1, 2, 3$ . Let  $U_i^{(n)}$  represent the interval between the  $n^{\text{th}}$  and the  $(n-1)^{\text{th}}$  event of type  $i$ . We assume that  $U_i^{(n)}$

has exponential distribution with parameter  $\lambda_i$ , independent of  $n$ . With an arbitrary time origin let us assume that  $U_i^{(1)}$ , the time to the first event of type  $i$ , has the same distribution as mentioned above. Further, let us assume that for fixed  $i$ , the distributions of  $U_i^{(n)}$  are independent for all  $n$ . Hence, we have three independent Poisson series of events and  $U_i$  has distribution  $\exp(\lambda_i)$ ,  $i = 1, 2, 3$  and  $U_1, U_2, U_3$  are mutually independent.

Let us now consider a system with two components in series. Let us assume that component 1 fails if either  $U_1$  or  $U_3$  happens, and component 2 fails if either  $U_2$  or  $U_3$  happens. The system fails if at least one component fails.

Let  $T_i$ ,  $i = 1, 2$ , be the time to failure of the  $i^{\text{th}}$  component. Then  $T_1 = \min(U_1, U_3)$ ,  $T_2 = \min(U_2, U_3)$  and

$$(16) \quad \begin{aligned} P[T_1 > s, T_2 > t] &= P[U_1 > s, U_2 > t, U_3 > \max(s, t)] \\ &= \exp[-\lambda_1 s - \lambda_2 t - \lambda_3 \max(s, t)]. \end{aligned}$$

This defines the distribution of a bivariate exponential distribution.

The reliability of the system at time  $t$  is

$$(17) \quad R(t) = P(T_1 > t, T_2 > t) = \exp[-(\lambda_1 + \lambda_2 + \lambda_3)t].$$

This shows that the time to failure of the system has exponential distribution with parameter  $\lambda_1 + \lambda_2 + \lambda_3$ . Also, the marginal distributions of  $T_1$  and  $T_2$  are exponential with parameters  $\lambda_1 + \lambda_3$  and  $\lambda_2 + \lambda_3$ , respectively.  $T_1$  and  $T_2$  are not independently distributed.

Let  $T = \min(T_1, T_2)$ . Then

$$(18) \quad P(T > t) = P(T_1 > t, T_2 > t) = \exp[-(\lambda_1 + \lambda_2 + \lambda_3)t].$$

(18) is really the same as (17).

Now suppose we have the same data as in section 2. We define  $\delta_j$ ,  $\tau_j$ ,  $z_j$ , and  $rV$  as in (5), (6), (7), and (8). Then  $z_1, \dots, z_r$  are independently exponentially distributed with parameter  $\lambda_1 + \lambda_2 + \lambda_3$ . With this modification we get that  $2rV(\lambda_1 + \lambda_2 + \lambda_3)$  has chi-square distribution with  $2r$  degrees of freedom and, as in (9) and (10), we get

$$(19) \quad P[e^{-(\lambda_1 + \lambda_2 + \lambda_3)t} > \exp \left\{ -t \cdot \frac{\chi_{2r}^2(\alpha)}{2rV} \right\}] = 1 - \alpha.$$

This gives an exact  $100(1-\alpha)$  per cent lower confidence bound for  $R(t)$ .

When  $n_1$  and  $n_2$  are different, we proceed as in the last part of section 2.

C. As in case B, but  $k > 2$ .

The bivariate exponential distribution has been generalized by Marshall and Olkin [9] to multivariate exponential distribution.

Let us consider a system with  $k$  components in series. Let  $S$  denote the set of vectors  $(s_1, \dots, s_k)$ , where each  $s_j = 0$  or  $1$ , but  $(s_1, \dots, s_k) \neq (0, \dots, 0)$ . For any vector  $s \in S$ ,  $\max(x_i s_i)$  is the maximum of the  $x_i$ 's for which  $s_i = 1$ . The multivariate exponential distribution is defined as

$$(20) \quad P(T_1 > x_1, \dots, T_k > x_k) = \exp \left[ - \sum_{s \in S} \lambda_s \max(x_i s_i) \right].$$

Here  $T_1, \dots, T_k$  denote the time to failure of components  $1, \dots, k$ . The marginal distribution of any subset of  $T_1, \dots, T_k$  is again a multivariate exponential distribution and, in particular, the one-dimensional marginal distributions are exponential and are not independent. As in case B, we can generate the distribution in (20) by considering  $2^k - 1$  independent

series of Poisson events and the associated failures of components. For example, for three components we need 7 series. The associated failures are

<u>s</u>	<u>shock</u>	<u>component</u>	<u>parameter</u>
1	1	1	$\lambda_1$
2	2	2	$\lambda_2$
3	3	3	$\lambda_3$
4	4	1,2	$\lambda_4$
5	5	1,3	$\lambda_5$
6	6	2,3	$\lambda_6$
7	7	1,2,3	$\lambda_7$

The reliability of the system at time  $t$  is given by

$$(21) \quad R(t) = \exp[- \sum_{s \in S} \lambda_s \cdot t].$$

As in case A, if we have data for the  $k$  components and define  $\delta_j^{(i)}$ ,  $j = 1, \dots, r$ ,  $i = 1, \dots, k$ , then  $z_j = \min_{i=1, \dots, k} \delta_j^{(i)}$ ,  $j = 1, \dots, r$ , has

exponential distribution with parameter  $\lambda \equiv \sum_{s \in S} \lambda_s$  and the method proposed under case A, section 3, is applicable.

D. When type II censoring with replacement.

Let us suppose that  $n$  components of a type are on test and we replace each failed component immediately by a new one. We stop as soon as  $r$  failures are obtained. This is termed in the literature as type II censoring of  $r$  out of  $n$  with replacement. Let  $D_j$  be defined as in (5). Let us define

$$(22) \quad \delta_j = nD_j, \quad j = 1, \dots, r.$$

Epstein [4] has shown that  $\delta_1, \dots, \delta_r$  are independently exponentially distributed with the same distribution as the parent population. With this new definition of  $\delta_j$  (and  $\tau_j, \delta_j^{(i)}$ ), the methods of sections 2 and 3 go through and we can obtain an exact  $100(1-\alpha)$  per cent lower confidence bound for the reliability of the system. As before, the method works when  $n_1, \dots, n_k$  are not all equal.

#### 4. Confidence Bounds Based on Maximum Likelihood Estimates

We will consider case A of section 3 and for illustration we will take  $k = 2$ .

Let us consider component  $i$  and let the order statistics obtained by type II censoring of  $r$  out of  $n$  without replacement be denoted by

$$x_{(1)}^{(i)} < x_{(2)}^{(i)} < \dots < x_{(r)}^{(i)}, \quad i = 1, 2.$$

Using the transformation given in (5), the likelihood of the sample is proportional to

$$L = \lambda_i^r \exp[-\lambda_i(\delta_1^{(i)} + \dots + \delta_r^{(i)})], \quad i = 1, 2,$$

where  $\delta_1^{(i)}, \dots, \delta_r^{(i)}$  are independently identically distributed as exponential with parameter  $\lambda_i$ . Hence, the maximum likelihood estimate of  $\lambda_i$  is given by

$$(23) \quad \hat{\lambda}_i = \frac{r}{\sum_{j=1}^r \delta_j^{(i)}}, \quad i = 1, 2.$$

Let

$$(24) \quad \delta = \sum_{j=1}^r \delta_j^{(1)}, \quad \tau = \sum_{j=1}^r \delta_j^{(2)}.$$

Then  $\delta$  and  $\tau$  have gamma distributions with density functions

$$f(\delta) = \frac{\lambda_1^r \delta^{r-1} e^{-\lambda_1 \delta}}{(r-1)!}, \quad \delta \geq 0, \lambda_1 > 0,$$

(25)

$$g(\tau) = \frac{\lambda_2^r \tau^{r-1} e^{-\lambda_2 \tau}}{(r-1)!}, \quad \tau \geq 0, \lambda_2 > 0.$$

Now,

$$\frac{\partial^2 \log L}{\partial \lambda_i^2} = -\frac{r}{\lambda_i^2}, \quad i = 1, 2.$$

$\frac{\lambda_i^2}{r}$  is, in a sense, the "asymptotic variance" of  $\hat{\lambda}_i$ . However, since  $\lambda_i$  is not known, we replace  $\frac{\lambda_i^2}{r}$  by  $\frac{\hat{\lambda}_i^2}{r}$ .

Definition: Let  $X_n$  be a random variable depending on  $n$  and let  $X$  be any other random variable. Then we define

$$X_n \xrightarrow{d} X \text{ as } n \rightarrow \infty$$

if, as  $n \rightarrow \infty$ , the cumulative distribution function of  $X_n$  tends to the cumulative distribution function of  $X$  at the points of continuity of the latter function. Also, we define  $X_n \xrightarrow{P} X$  if, given any  $\epsilon > 0$ ,

$$P(|X_n - X| \geq \epsilon) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

We shall prove

Theorem 4.1:

Let  $\delta_1, \dots, \delta_r$  and  $\tau_1, \dots, \tau_r$  be two independent random samples from exponential distributions with parameters  $\lambda_1$  and  $\lambda_2$ , respectively.

Then

$$T_r = \frac{\frac{r}{\delta} + \frac{r}{\tau} - \lambda_1 - \lambda_2}{\sqrt{\frac{r}{\delta^2} + \frac{r}{\tau^2}}} \xrightarrow{d} N(0,1) \text{ as } r \rightarrow \infty,$$

where  $\delta = \sum_{i=1}^r \delta_i^{(1)}$  and  $\tau = \sum_{i=1}^r \delta_i^{(2)}$ .

We will require the following theorem to prove Theorem 4.1:

Theorem 4.2:

Suppose  $(x_{1\xi}, \dots, x_{k\xi}, \xi = 1, \dots, r)$  is a sample from a  $k$ -dimensional distribution with finite means  $\{\mu_i\}$  and positive definite covariance matrix  $\|\sigma_{ij}\|$ ,  $i, j = 1, \dots, k$ . Let  $g(x_1, \dots, x_k)$  be a function which possesses first derivatives  $\frac{\partial g}{\partial x_i} = g_i$ , say,  $i = 1, \dots, k$ , at all points in some neighborhood of  $(\mu_1, \dots, \mu_k)$ , and let  $g_i^0 = g_i(\mu_1, \dots, \mu_k)$ . Then if at least one of the  $g_i^0$  is  $\neq 0$ ,

$$\sqrt{r}[g(\bar{x}_1, \dots, \bar{x}_k) - g(\mu_1, \dots, \mu_k)]$$

$$\xrightarrow{d} N\left[0, \sum_{i,j=1}^k \sigma_{ij} g_i^0 g_j^0\right] \text{ as } r \rightarrow \infty.$$

Proof: See [14, p. 260, section 9.3.1a].

Proof of Theorem 4.1:

Let  $\bar{\delta} = \delta/r$ ,  $\bar{\tau} = \tau/r$ . From what we are given,

$$E(\delta_i) = \frac{1}{\lambda_1}, \quad V(\delta_i) = \frac{1}{\lambda_1^2},$$

$$E(\tau_j) = \frac{1}{\lambda_2}, \quad V(\tau_j) = \frac{1}{\lambda_2^2},$$

$$\text{Cov}(\delta_i, \tau_j) = 0; \quad i, j = 1, \dots, r.$$



Consider  $g(x,y) = \frac{\frac{1}{x} + \frac{1}{y} - \lambda_1 - \lambda_2}{\sqrt{\frac{1}{x^2} + \frac{1}{y^2}}}.$

Then

$$g_1^0 = \frac{\partial g}{\partial x} \Big|_{x=\frac{1}{\lambda_1}, y=\frac{1}{\lambda_2}} = - \frac{\lambda_1^2}{(\lambda_1^2 + \lambda_2^2)^{1/2}} \neq 0, \pm \infty,$$

and

$$g_2^0 = \frac{\partial g}{\partial y} \Big|_{x=\frac{1}{\lambda_1}, y=\frac{1}{\lambda_2}} = - \frac{\lambda_2^2}{(\lambda_1^2 + \lambda_2^2)^{1/2}} \neq 0, \pm \infty.$$

Hence, by Theorem 4.2, as  $r \rightarrow \infty$ ,

$$\sqrt{r}[g(\bar{\delta}, \bar{\tau}) - g(\frac{1}{\lambda_1}, \frac{1}{\lambda_2})]$$

$$\xrightarrow{d} N(0, \sum_{i,j=1}^2 \sigma_{ij} g_i^0 g_j^0).$$

Now

$$g(\bar{\delta}, \bar{\tau}) = \frac{\frac{r}{\bar{\delta}} + \frac{r}{\bar{\tau}} - \lambda_1 - \lambda_2}{\sqrt{\frac{r^2}{\bar{\delta}^2} + \frac{r^2}{\bar{\tau}^2}}},$$

$$g(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}) = 0,$$

and

$$\sum_{i,j=1}^2 \sigma_{ij} g_i^0 g_j^0 = \frac{1}{\lambda_1^2} \cdot \frac{\lambda_1^4}{(\lambda_1^2 + \lambda_2^2)} + \frac{1}{\lambda_2^2} \cdot \frac{\lambda_2^4}{(\lambda_1^2 + \lambda_2^2)} = 1.$$

Hence,

$$\begin{aligned} & \sqrt{r}[g(\bar{\delta}, \bar{\tau}) - g(\frac{1}{\lambda_1}, \frac{1}{\lambda_2})] \\ &= \frac{\frac{r}{\bar{\delta}} + \frac{r}{\bar{\tau}} - \lambda_1 - \lambda_2}{\sqrt{\frac{r^2}{\bar{\delta}^2} + \frac{r^2}{\bar{\tau}^2}}} \xrightarrow{d} N(0,1), \quad \text{as } r \rightarrow \infty, \end{aligned}$$

and the theorem is proved.

Theorem 4.1 gives us an asymptotically exact confidence interval for the reliability of the system. Let  $k_\alpha$  be such that

$$\int_{-\infty}^{-k_\alpha} \frac{1}{\sqrt{2\pi}} e^{-1/2t^2} dt = \alpha.$$

Then

$$P\left[\frac{\frac{r}{\delta} + \frac{r}{\tau} - \lambda_1 - \lambda_2}{\sqrt{\frac{r}{\delta^2} + \frac{r}{\tau^2}}} > -k_\alpha\right] = 1-\alpha,$$

$$(26) \quad P[\lambda_1 + \lambda_2 < \frac{r}{\delta} + \frac{r}{\tau} + k_\alpha \sqrt{\frac{r}{\delta^2} + \frac{r}{\tau^2}}] = 1-\alpha,$$

$$(27) \quad P[e^{-(\lambda_1 + \lambda_2)t} > \exp\left\{-t\left(\frac{r}{\delta} + \frac{r}{\tau} + k_\alpha \sqrt{\frac{r}{\delta^2} + \frac{r}{\tau^2}}\right)\right\}] = 1-\alpha.$$

(27) gives the 100(1- $\alpha$ ) per cent lower confidence bound for the reliability. The bound is exact if  $r \rightarrow \infty$ . The method of this section generalizes when the number of components in series is more than 2.

Since this confidence interval (lower confidence bound, 1) is based on the maximum likelihood estimate, for large samples it is the smallest confidence interval. It is not known how good this bound is for small samples.

We will make some comparison of this method with the method of section 2.

When the number of items on test for each component is different, the remark at the end of section 2 applies.

##### 5. Kraemer's Method [7]

For the case A of section 3, kraemer's first method is as follows:

Let  $a_i = \frac{1}{\lambda_i}$ ,  $i = 1, 2, \dots, k$ .

$$\hat{a}_i = \left[ \sum_{j=1}^{r_i} X_{(j)}^{(i)} + (n_i - r_i) X_{(r_i)}^{(i)} \right] / r_i.$$

$\frac{2r_i \hat{a}_i}{a_i}$  is distributed as chi-square with  $2r_i$  degrees of freedom.

Let  $A = \min_{i=1,2,\dots,k} (r_i \hat{a}_i)$ . Then

$$P[e^{-(\lambda_1 + \dots + \lambda_k)t} > \exp\left\{-t \frac{\chi_{2(r_1 + \dots + r_k)}^2(\alpha)}{2A}\right\}] \geq 1 - \alpha,$$

where, if  $U$  is a chi-square variable with  $2(r_1 + \dots + r_k)$  degrees of freedom, then  $\chi_{2(r_1 + \dots + r_k)}^2(\alpha)$  is defined as

$$P[U < \chi_{2(r_1 + \dots + r_k)}^2(\alpha)] = 1 - \alpha.$$

## 6. A Comparison of the Methods

We will compare the methods of sections 2, 4, and 5 for a system with two independent components in series. The time to failure of each component is distributed exponentially with parameter  $\lambda_i$ ,  $i = 1, 2$ . The comparison will be based on the expected values of the upper confidence bounds for  $\lambda_1 + \lambda_2$  under each method. Since the left-hand end point is zero, this gives the expected length of the confidence interval for  $\lambda_1 + \lambda_2$ . Since the reliability is a (strictly) monotone function of  $\lambda_1 + \lambda_2$ , this furnishes a valid comparison.

### Exact Method of Section 2

$$P[\lambda_1 + \lambda_2 < \frac{\chi_{2r}^2(\alpha)}{2rv}] = 1 - \alpha.$$

Since  $z_i$  are independent, identical exponential variables with parameter  $\lambda_1 + \lambda_2$ , by (8)  $rV$  has gamma distribution and

$$(28) \quad E\left[\frac{\chi_{2r}^2(\alpha)}{2rV}\right] = \frac{\chi_{2r}^2(\alpha)}{2} E\left[\frac{1}{\sum_{i=1}^r z_i}\right] = \frac{\chi_{2r}^2(\alpha)}{2} \cdot \frac{\lambda_1 + \lambda_2}{r-1}$$

where

$$\begin{aligned} E\left[\frac{1}{\sum_{i=1}^r z_i}\right] &= \int_0^\infty \frac{1}{x} \cdot \frac{(\lambda_1 + \lambda_2)^r x^{r-1} e^{-(\lambda_1 + \lambda_2)x}}{(r-1)!} dx \\ &= \frac{\lambda_1 + \lambda_2}{r-1} \int_0^\infty \frac{(\lambda_1 + \lambda_2)^{r-1} x^{r-2} e^{-(\lambda_1 + \lambda_2)x}}{(r-2)!} dx \\ &= \frac{\lambda_1 + \lambda_2}{r-1}, \text{ provided } r \geq 2. \end{aligned}$$

#### Kraemer's Method of Section 5

$$P[\lambda_1 + \lambda_2 < \frac{\chi_{4r}^2(\alpha)}{2A}] \geq 1-\alpha,$$

where

$$\begin{aligned} A &= \min(r\hat{a}_1, r\hat{a}_2) \\ &= \min\left[\frac{1}{2\lambda_1} \cdot 2r\hat{a}_1\lambda_1, \frac{1}{2\lambda_2} \cdot 2r\hat{a}_2\lambda_2\right]. \end{aligned}$$

Since  $2r\hat{a}_i\lambda_i$  are distributed as  $\chi^2$  with  $2r$  degrees of freedom and are independent for  $i = 1, 2$ ,

$$\begin{aligned} P(A > x) &= P(2r\hat{a}_1\lambda_1 > 2\lambda_1 x) P(2r\hat{a}_2\lambda_2 > 2\lambda_2 x) \\ &= \left[ \int_{\lambda_1 x}^\infty \frac{1}{(r-1)!} u^{r-1} e^{-u} du \right] \left[ \int_{\lambda_2 x}^\infty \frac{1}{(r-1)!} v^{r-1} e^{-v} dv \right]. \end{aligned}$$

Hence, the probability density function of  $A$  is given by

$$f(x) = \frac{\lambda_1^r x^{r-1} e^{-\lambda_1 x}}{(r-1)!} \int_x^\infty \frac{\lambda_2^r v^{r-1} e^{-\lambda_2 v}}{(r-1)!} dv$$

$$+ \frac{\lambda_2^r x^{r-1} e^{-\lambda_2 x}}{(r-1)!} \int_x^\infty \frac{\lambda_1^r v^{r-1} e^{-\lambda_1 v}}{(r-1)!} dv, \quad x \geq 0.$$

Integrating by parts,

$$f(x) = \frac{\lambda_1^r x^{r-1} e^{-\lambda_1 x}}{(r-1)!} \sum_{s=0}^{r-1} e^{-\lambda_2 x} \frac{(\lambda_2 x)^s}{s!}$$

$$+ \frac{\lambda_2^r x^{r-1} e^{-\lambda_2 x}}{(r-1)!} \sum_{s=0}^{r-1} e^{-\lambda_1 x} \frac{(\lambda_1 x)^s}{s!}.$$

Hence,

$$(29) \quad E\left(\frac{1}{A}\right) = \int_0^\infty \frac{1}{x} f(x) dx = \sum_{s=0}^{r-1} \frac{(r+s-2)!}{(r-1)! s!} \frac{\lambda_1^r \lambda_2^s + \lambda_2^r \lambda_1^s}{(\lambda_1 + \lambda_2)^{r+s-1}}, \quad \text{for } r \geq 2,$$

and

$$(30) \quad E\left(\frac{x_{4r}^2(\alpha)}{2A}\right) = \frac{x_{4r}^2(\alpha)}{2} \sum_{s=0}^{r-1} \frac{(r+s-2)!}{(r-1)! s!} \frac{\lambda_1^r \lambda_2^s + \lambda_2^r \lambda_1^s}{(\lambda_1 + \lambda_2)^{r+s-1}}, \quad \text{provided } r \geq 2.$$

#### Asymptotic Method of Section 4

$$P[\lambda_1 + \lambda_2 < \frac{r}{\delta} + \frac{r}{\tau} + k_\alpha \sqrt{\frac{r}{\delta^2} + \frac{r}{\tau^2}}] = 1 - \alpha.$$

Using (25),

$$(31) \quad E\left(\frac{r}{\delta} + \frac{r}{\tau}\right) = \frac{r}{r-1} (\lambda_1 + \lambda_2).$$

Substituting  $\delta = \rho \cos \theta$ ,  $\tau = \rho \sin \theta$ , and using (25), we get

$$E\left[\sqrt{\frac{r}{\delta^2} + \frac{r}{\tau^2}}\right] = E\left[\frac{\sqrt{r} \sqrt{\delta^2 + \tau^2}}{\delta \tau}\right]$$

$$= \frac{\sqrt{r} (\lambda_1 \lambda_2)^{r(2r-2)!}}{(r-1)! (r-1)!} \int_0^{\pi/2} \frac{(\cos \theta \sin \theta)^{r-2} d\theta}{(\lambda_1 \cos \theta + \lambda_2 \sin \theta)^{2r-1}},$$

provided  $r \geq 2$ .

(Put  $t = \tan \frac{\theta}{2}$ )

$$= \frac{\sqrt{r} (\lambda_1 \lambda_2)^r (2r-2)!}{(r-1)! (r-1)!} \cdot \frac{2^{r-1}}{\lambda_1^{2r-1}} \int_0^1 \frac{t^{r-2} (1-t^2)^{r-2} (1+t^2)^2 dt}{(a+t)^{2r-1} (b-t)^{2r-1}}$$

where

$$a = \sqrt{1 + \left(\frac{\lambda_2}{\lambda_1}\right)^2} - \frac{\lambda_2}{\lambda_1} > 0,$$

$$b = \sqrt{1 + \left(\frac{\lambda_2}{\lambda_1}\right)^2} + \frac{\lambda_2}{\lambda_1} > 1.$$

We will break the integrand into partial fractions:

$$\frac{t^{r-2} (1-t^2)^{r-2} (1+t^2)^2}{(a+t)^{2r-1} (b-t)^{2r-1}} = \sum_{i=1}^{2r-1} \frac{A_i}{(a+t)^i} + \sum_{j=1}^{2r-1} \frac{B_j}{(b-t)^j},$$

or

$$\begin{aligned} & \sum_{s=0}^{r-2} \binom{r-2}{s} (-1)^s [t^{r+2s-2} + 2t^{r+2s} + t^{r+2s+2}] \\ &= \sum_{i=1}^{2r-1} A_i (a+t)^{2r-1-i} (b-t)^{2r-1} \\ &+ \sum_{j=1}^{2r-1} B_j (b-t)^{2r-1-j} (a+t)^{2r-1}, \end{aligned}$$

where A's and B's are independent of  $t$ .

We have  $4r-2$  unknowns. We will differentiate  $n$  times ( $n = 0, 1, \dots, 2r-2$ ) with respect to  $t$  and put  $t = -a$  and  $t = b$ . This will give us two systems of  $2r-1$  equations each. Each system is triangular and we can easily solve sequentially for  $A_{2r-1}, B_{2r-1}, \dots, A_1, B_1$ .

For the  $n^{\text{th}}$  differentiation the general pair of equations is

$$\begin{aligned}
& \sum_{s=0}^{r-2} \binom{r-2}{s} (-1)^s \left[ \binom{r+2s-2}{n} (-a)^{r+2s-2-n} \right. \\
& \quad \left. + 2 \binom{r+2s}{n} (-a)^{r+2s-n} + \binom{r+2s+2}{n} (-a)^{r+2s+2-n} \right] \\
& = \sum_{k=0}^n A_{2r-1-k} \binom{2r-1}{n-k} (a+b)^{2r-1-(n-k)} (-1)^{n-k}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{s=0}^{r-2} \binom{r-2}{s} (-1)^s \left[ \binom{r+2s-2}{n} b^{r+2s-2-n} \right. \\
& \quad \left. + 2 \binom{r+2s}{n} b^{r+2s-n} + \binom{r+2s+2}{n} b^{r+2s+2-n} \right] \\
& = \sum_{k=0}^n B_{2r-1-k} \binom{2r-1}{n-k} (a+b)^{2r-1-(n-k)} (-1)^k
\end{aligned}$$

with the proviso that  $\binom{p}{q} = 0$  if  $p < q$ . After solving for A's and B's, we get, using (31),

$$\begin{aligned}
(32) \quad & E\left[\frac{r}{8} + \frac{r}{\tau} + k_{\alpha} \sqrt{\frac{r}{8} 2^{\frac{r}{\tau}}}\right] \\
& = \frac{r}{r-1} (\lambda_1 + \lambda_2) + k_{\alpha} \frac{\sqrt{r} (\lambda_1 \lambda_2)^r (2r-2)! 2^{r-1}}{(r-1)! (r-1)! \lambda_1^{2r-1}} \times \\
& \quad \left[ \sum_{i=2}^{2r-1} \frac{A_i}{i-1} \left( \frac{1}{a^{i-1}} - \frac{1}{(a+1)^{i-1}} \right) + A_1 \log \left( \frac{a+1}{a} \right) \right. \\
& \quad \left. + \sum_{j=2}^{2r-1} \frac{B_j}{j-1} \left( \frac{1}{(b-1)^{j-1}} - \frac{1}{b^{j-1}} \right) + B_1 \log \left( \frac{b}{b-1} \right) \right], \\
& \quad \text{for } r \geq 2.
\end{aligned}$$

We will now make some numerical comparisons between (28), (30), and (32).

### Numerical Comparisons

The following table gives some numerical values of  $E$  (upper 95 per cent confidence bound for  $\lambda_1 + \lambda_2$ ) based on (28), (30), and (32). In (30) we have used tables of cumulative binomial distribution as follows:

Let

$$B(n, k, p) = \sum_{s=k}^n \binom{n}{s} p^s (1-p)^{n-s}.$$

Then

$$\begin{aligned} & \sum_{s=0}^{r-1} \frac{(r+s-2)!}{(r-1)!s!} \frac{\lambda_1^r \lambda_2^s + \lambda_2^r \lambda_1^s}{(\lambda_1 + \lambda_2)^{r+s-1}}, \quad r \geq 2 \\ &= \frac{\lambda_1}{r-1} \sum_{s=0}^{r-1} \binom{r+s-2}{r-2} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^{r-1} \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^s \\ & \quad + \frac{\lambda_2}{r-1} \sum_{s=0}^{r-1} \binom{r+s-2}{r-2} \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{r-1} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^s \\ &= \frac{\lambda_1}{r-1} \sum_{s=r-1}^{2r-2} \binom{2r-2}{s} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^s \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{2r-2-s} \\ & \quad + \frac{\lambda_2}{r-1} \sum_{s=r-1}^{2r-2} \binom{2r-2}{s} \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^s \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^{2r-2-s} \\ &= \frac{\lambda_1}{r-1} B(2r-2, r-1, \frac{\lambda_1}{\lambda_1 + \lambda_2}) \\ & \quad + \frac{\lambda_2}{r-1} \left\{ 1 - B(2r-2, r, \frac{\lambda_1}{\lambda_1 + \lambda_2}) \right\}. \end{aligned}$$

In (32), instead of using the partial fraction expansion we have performed numerical integration of

$$\int_0^{\pi/2} \frac{(\cos \theta \sin \theta)^{r-2} d\theta}{(\lambda_1 \cos \theta + \lambda_2 \sin \theta)^{2r-1}}.$$



We have used Simpson's rule after subdividing the interval  $[0, \pi/2]$  into ten equal parts.

$\lambda_1$	$\lambda_2$	$r$	E (exact)	E (Kraemer)	E (asymptotic)
2	2.5	5	10.30	11.37	8.72
		10	7.85	8.40	6.90
		20	6.60	6.98	*
		30	6.14	6.48	*
1	5	5	13.73	19.66	12.20
		10	10.47	15.49	10.47
		20	8.80	13.41	*
		30	8.18	12.64	*
$\lambda$	$\lambda$	5	$4.58\lambda$	$5.00\lambda$	$3.87\lambda$
		10	$3.49\lambda$	$3.67\lambda$	$3.06\lambda$
		20	$2.94\lambda$	$3.03\lambda$	*
		30	$2.73\lambda$	$2.79\lambda$	*

\* For larger values of  $r$  the error involved in approximate integration becomes large and hence these values have not been given.

As has been pointed out before, the above table also gives a valid comparison between the confidence intervals of  $R(t)$  as given by the different methods.

In the range of  $\alpha$  that we are interested in (e.g.,  $1-\alpha = .90, .95$ , etc), it is expected that the exact method will give a shorter confidence interval than the Kraemer method. Since Kraemer's method is an

approximate one, in general, for  $\lambda_1$  and  $\lambda_2$  widely different (as in the second set of examples), the exact method is expected to be significantly superior to Kraemer's. This happens because the error of approximation in her method exceeds the precision obtained by using a larger number of degrees of freedom in the chi-square distribution. When  $\lambda_1$  and  $\lambda_2$  are close (as in the first set of examples), sometimes Kraemer's method may give a better confidence interval than the exact method; but the difference between the two methods is not expected to be large. For very small values of  $\alpha$  (e.g.,  $1-\alpha = .99, .999$ , etc.), there is some numerical evidence that sometimes Kraemer's method is slightly better than the exact method when  $\lambda_1 = \lambda_2$ . It may be pointed out that for  $\lambda_1 = \lambda_2$ , the difference between the expected value of the upper confidence bound for  $\lambda_1 + \lambda_2$  given by Kraemer's method, and the expected value as given by the exact method, goes to zero as  $r$  goes to infinity.

Kraemer's method has one advantage over the exact method. The former is applicable even when the termination points  $(r_i)$  for the components are different. For the exact method some data have to be ignored to equalize the termination points.

For large  $r$  it is expected that the asymptotic method will be better than the exact method. Without a knowledge of the rate of convergence of the exact and asymptotic methods, it is not possible to assert to any degree of precision how the two methods will compare. However, it is expected that the exact method will compare favorably with the asymptotic method for large  $r$ .

Since the methods of [1] and [8] are based on inverting uniformly most powerful unbiased tests, they may give better lower confidence bounds

than those given by the method proposed in this paper. This is especially true for small values of  $r$ . For large values of  $r$  the confidence bounds given in [1] and [8] will tend to the bounds given by the method of [2], and will be the same as those given by the asymptotic method. As we have already pointed out, the methods of [1] and [8] are computationally complicated even for the case of two components, and work only when the components are independent. In fact, the computational difficulties motivated the authors to get (approximately) asymptotic results in their subsequent joint paper [2]. These three methods, however, do not require the same termination points for every component.

When  $\lambda_2 \rightarrow 0$ , the exact method and the asymptotic method proposed in this paper can be easily compared. If  $\lambda_2 \rightarrow 0$ , then we are reduced to the case of a single component with exponential time to failure distribution. In this case,  $\tau_i = \infty$  for all  $i$ , and

$$z_i = \min(\delta_i, \tau_i) = \delta_i$$

$$rV = \sum_{i=1}^r \delta_i.$$

(10) reduces to

$$P[e^{-\lambda_1 t} > \exp(-t \cdot \frac{\chi_{2r}^2(\alpha)}{2rV})] = 1 - \alpha,$$

the well-known result about a single component.

Let  $\lambda_2 \rightarrow 0$  in (27). Then

$$P[e^{-\lambda_1 t} > \exp\left\{-t\left(\frac{r}{\delta} + k_\alpha \frac{\sqrt{r}}{\delta}\right)\right\}] = 1 - \alpha.$$

Following Halperin [5],

$$\sqrt{r}(R(t) - \hat{R}(t))\sigma_u \xrightarrow{L} N(0,1).$$

where

$$\sigma(u) = - \frac{1}{R(t) \log R(t)},$$

$$\hat{R}(t) = e^{-\hat{\lambda}_1 t},$$

$$\hat{\lambda}_1 = \frac{r}{\delta}.$$

If we replace  $\sigma_u$  by  $\hat{\sigma}_u = - \frac{1}{\hat{R}(t) \log \hat{R}(t)}$ , then the lower confidence bound for  $R(t)$  becomes

$$\exp\left(-t \cdot \frac{r}{\delta}\right) \left\{ 1 - \frac{k_\alpha t \sqrt{r}}{\delta} \right\}.$$

By the weak law of large numbers (see [10, p. 92]),  $\frac{\delta}{r} \xrightarrow{P} \frac{1}{\lambda_1}$  as  $r \rightarrow \infty$ . Then  $\frac{r}{\delta} \xrightarrow{P} \lambda_1$  and

$$\frac{\sqrt{r}}{\delta} = \frac{1}{\sqrt{r}} \cdot \frac{r}{\delta} \xrightarrow{P} 0.$$

Hence, asymptotically  $\exp\left(-\frac{k_\alpha t \sqrt{r}}{\delta}\right)$  and  $1 - \frac{k_\alpha t \sqrt{r}}{\delta}$  are the same.

This checks our asymptotic result for a single component against the well-known result as given above.

Finally, it is easily shown that the exact lower confidence bound converges in probability to the true reliability as  $r$  approaches infinity

From (10) we note that the lower confidence bound for  $R(t)$  is  $\exp\left\{-t \cdot \frac{x_{2r}^2(\alpha)}{2rV}\right\}$ . By (18),  $V$  is the arithmetic average of independent identically distributed random variables. Hence, by the weak law of large numbers,

$$V \xrightarrow{P} \frac{1}{\lambda_1 + \lambda_2} \text{ as } r \rightarrow \infty.$$

Also,

$$\frac{x_{2r}^2(\alpha)}{2r} \rightarrow 1 \text{ as } r \rightarrow \infty.$$

Hence, as  $r \rightarrow \infty$ ,

$$\exp\left\{-t \cdot \frac{\chi_{2r}^2(\alpha)}{2rV}\right\} \xrightarrow{P} e^{-(\lambda_1 + \lambda_2)t} = R(t).$$

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13. **ABSTRACT:** Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as (TS), (S), (C), or (U).

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. **KEY WORDS:** Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.